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impc2014.org | impc@impc2014.org | +56 2 2652 1528

ABSTRACT 211

Rating Cumulative Grade Recovery Curve Data

Alex van der Spek
Consultant, ZDoor BV, The Netherlands

Christian O'Keefe*
Chief Technology Officer, CiDRA Minerals Processing, USA

ABSTRACT

Evaluation of inequalities of the partial sums defining cumulative grade and recovery prove that cumulative grade has an upper bound not dependent on the bank number and cumulative recovery possesses an upper bound proportional to the bank number. This explains one observed characteristic of cumulative grade versus cumulative recovery curves.

The existence of such maxima combined with the observed shape of cumulative grade-recovery curves leads us to postulate that these curves can be well described by rectangular hyperbolae. Rectangular hyperbolae are characterized by asymptotes at right angles to each other. It is proven that this property can be used to derive an equation for the cumulative mass flow of the concentrate in terms of the asymptotic values of cumulative grade and cumulative recovery.

Fitting hyperbolae to experimental cumulative grade cumulative recovery plant data then results in estimates of the maximum grade, the maximum recovery and a constant which may be taken as a performance coefficient of a row.

Normalization of the cumulative grade and the cumulative recovery by their respective maxima gives rise to a graphical representation that can be used to evaluate the operating performance of a row of floatation bank cells. The normalization procedure removes the bias that is introduced by inevitable variation in mass flow of the feed and variation in feed mineralogy or feed particle size distribution.

Reduction of the normalized data collapses all cumulative grade-recovery data onto one and the same curve. This removes the differences caused by the variation in row performance. The reduced cumulative grade-recovery data scatters around a common hyperbola. This can be used to gauge the validity of the assumptions made.

Normalized cumulative mass pull is determined from the fitted values of the maximum recovery and maximum grade. This may serve as an alternative performance indicator of the floatation row.

INTRODUCTION

In flotation based minerals processing the separation of the valuable minerals from the gangue is almost exclusively performed using flotation cells aligned in rows. In each row the tails (underflow) of a cell feed the next one down the row. These flotation cells may be stand alone or grouped into banks of flotation cells. The latter arrangement uses a common froth depth control based on the level in the last cell in the bank. In addition, this arrangement makes it difficult to obtain accurate tails samples flowing from one cell to the next cell in a bank. As a result, we will treat the cells in a bank as a single entity, and for the purposes of this paper, the term bank will be used to refer to either a single flotation cell or a bank of flotation cells. In each row the concentrate is typically collected in a common launderer for the rougher section and a separate common launder for the scavenger section. Analysis of the performance of a row using the cumulative grade and the cumulative recovery drives one to separate the rougher section of a row from the scavenger section, treating each as if they were separate rows, with the feed of the scavenger section simply being the tails of the rougher. Therefore, references to a row imply either a rougher section or a scavenger section. A row of single cell banks is shown for clarification in Figure 1.

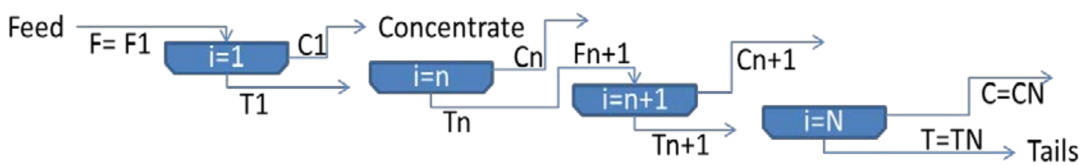

Figure 1 A row of flotation cells

It is of interest, of course, to investigate the performance of such a row of banks. The common quantities to measure performance are the cumulative recovery of valuable material and the cumulative grade of the concentrate. Experience shows that cross plotting cumulative grade versus cumulative recovery always shows curves, which are shaped much like hyperbolae. That is to say, there appears to be a maximum grade and a maximum recovery given the same operating conditions, constant feed grade and constant mass flow of the feed. It remains difficult, though, to compare performance when constant feed grade and constant mass flow of the feed cannot be achieved. This is so because the maximum grade of the concentrate is defined by the mineralogy of the feed, i.e. by the mass fraction of floatable valuable material therein. Likewise, the maximum recovery is, amongst others, defined by the volumetric flow rate of the feed as this determines, given constant volume of the cells, the residence time in each cell. The liberation size and particle size distribution of the feed also has an impact on the recovery, grade or both.

We will present a normalization procedure by which the raw cumulative grade and raw cumulative recovery data can be brought into a common form that will enable easy comparison of bank performance in a row. The common, normalized form is unbiased as it removes the influence of variations of mineralogy and mass flow of the feed.

The normalization procedure assumes that the shape of the cumulative grade versus cumulative recovery curves resemble hyperbolae with grade asymptotic maxima and recovery asymptotic maxima. We present a reduction procedure by which the normalized grade recovery curves collapse into one common shape independent of both the variations in feed and any variations in row performance or row operating conditions. This reduced form can be used to gauge the assumption made.

The theoretical underpinning of the normalization and reduction procedure rests on the evaluation of inequalities of the partial sums by which cumulative recovery and cumulative grade are defined. This can be done without making any assumptions about the single cell performance. The only requirement is that each bank cannot produce a grade higher than a certain maximum and that each bank cannot recover more than a certain maximum. These two assumptions are enough to explain and prove an experimentally observed characteristic of cumulative grade versus cumulative recovery curves; the cumulative recovery increases down a row with increasing bank number but cumulative grade does not.

Single bank recovery and grade equations

In a paper by Neethling and Cilliers (2012), equations for single cell performance were derived that have closed form solutions for grade and mass pull. These equations can be reworked easily into closed form solutions for grade and recovery. The equation for grade is simply a function of the tails grade:

$$G_C = \frac{G_f(G_T - G_{nf}) + bG_T(G_f - G_{nf})}{(G_T - G_{nf}) + b(G_{mf} - G_{nf})} \quad (1)$$

Where G_C is the grade of the concentrate, G_T is the grade of the tails of this cell, b is a constant dependent on the cell operating parameters. G_{nf} is the grade of the non-floatable valuable material and G_f is the grade of the floatable material. G_{nf} encompasses material that will not float due to particle size, liberation class, or chemical composition. It is easy to show that the maximum value that the grade of the concentrate can assume equals G_f whereas the minimum is G_{nf}. For the recovery we find:

$$R = \left(1 - \frac{G_T}{G_F}\right) + \frac{G_T}{G_F} a \left(b + \frac{G_T - G_{nf}}{G_f - G_{nf}}\right) \quad (2)$$

Where a is a constant dependent on the mass feed and G_F is the feed grade of this cell incorporating both the floatable feed grade, G_f, and the non-floatable feed grade, G_{nf}. The recovery likewise has an upper and a lower bound. The maximum recovery is obtained when the tails grade equals the grade of the non-floatable material:

$$R_{\max} = \left(1 - \frac{G_{nf}}{G_F}\right) + \frac{G_{nf}}{G_F} ab \quad (3)$$

The derivation of the above formula for the purpose of this paper is not important. All that matters is that for single cell or bank behaviour there are upper bounds to both the grade and the recovery.

METHODOLOGY

Cumulative grade cumulative recovery curves

A plot of the cumulative grade down the banks in a row of flotation cells versus the cumulative recovery often shows a characteristic shape that resembles the shape of a hyperbola. The cumulative grade drops sharply as the cumulative recovery increases to what appears to be a limit. Conversely when the cumulative recovery decreases the cumulative grade increases slowly but also appears to possess an upper bound. If the shape of the cumulative grade versus cumulative recovery curve is indeed a hyperbola then the upper bounds of the cumulative grade and cumulative recovery are the asymptotes of a hyperbola. The theory presented here does not make any assumptions about the single cell grade or single cell recovery other than that both single cell grade and single cell recovery possess an upper bound.

Cumulative recovery sum and inequalities

We define the cumulative recovery $R_\Sigma(n)$ as per the cumulative sum of the mass flow of target mineral or element recovered in the launder or concentrate from individual banks 1 to *N* in a row of *N* banks over the total mass flow of target mineral or element in the feed:

$$R_\Sigma(n) = \frac{\sum_{i=1}^{n\le N} m_{Ci} G_{Ci}}{m_F G_F} \tag{4}$$

Where m is the mass flow of solids, G is grade, the subscript Ci denotes the concentrate of bank i and the subscript F denotes the row feed. The recovery of a single bank R_i is similarly defined as:

$$R_i = \frac{m_{Ci} G_{Ci}}{m_{Fi} G_{Fi}} \tag{5}$$

Where the denominator is the mass feed m and grade G into the bank number i. By combination of the above two equations, we find that cumulative recovery is a weighted sum over the individual grades as per the equation below.

$$R_\Sigma(n) = \sum_{i=1}^{n\le N} R_i \left(\frac{m_{Fi}}{m_F}\right)\left(\frac{G_{Fi}}{G_F}\right) \tag{6}$$

If we now realize that the feed to each bank must be smaller than or equal to the feed to the first bank and equivalently the grade of the feed to each bank is smaller than or equal to the feed grade, we can derive an inequality for the cumulative recovery:

$$R_\Sigma(n) = \sum_{i=1}^{n\le N} R_i \left(\frac{m_{Fi}}{m_F}\right)\left(\frac{G_{Fi}}{G_F}\right) \le \hat{R}\sum_{i=1}^{n\le N} \left(\frac{m_{Fi}}{m_F}\right)\left(\frac{G_{Fi}}{G_F}\right) \le \hat{R}\sum_{i=1}^{n\le N} 1 = n\hat{R} \le N\hat{R} \tag{7}$$

Where $\hat{R}$ is the maximum recovery attainable in a single bank. Thus an absolute maximum to the cumulative recovery is given by the above equation as:

$$R_{\Sigma}(n) \leq N\hat{R} = R_{\max} \tag{8}$$

Thus the intuitive notion that cumulative recovery increases with bank number is substantiated by the above inequality. The cumulative recovery of n banks is always smaller than the cumulative recovery of (n+1) banks. Cumulative recovery is thus a monotonically increasing function of the bank number.

Cumulative grade sum and inequalities

We define the cumulative grade G_{Σ} as the cumulative weighted sum over the grades of each individual bank numbered i:

$$G_{\Sigma}(n) = \frac{\sum_{i=1}^{n\leq N} m_{Ci}G_{Ci}}{\sum_{i=1}^{n\leq N} m_{Ci}} = \frac{\sum_{i=1}^{n\leq N} m_{Ci}G_{Ci}}{M_{\Sigma}(n)} \tag{9}$$

In this equation the G_{Ci} is the grade of the concentrate in bank i. M_{Σ} is the cumulative mass flow in the concentrate. Let $\hat{G}$ be the maximum attainable grade in a single bank, then the following inequality holds:

$$G_{\Sigma}(n) = \frac{\sum_{i=1}^{n\leq N} m_{Ci}G_{Ci}}{\sum_{i=1}^{n\leq N} m_{Ci}} \leq \hat{G}\frac{\sum_{i=1}^{n\leq N} m_{Ci}}{\sum_{i=1}^{n\leq N} m_{Ci}} = \hat{G} \tag{10}$$

Thus, an absolute maximum of the cumulative grade is given by:

$$G_{\Sigma}(n) \leq \hat{G} = G_{\max} \tag{11}$$

This proves that cumulative grade is independent of the bank number. The cumulative grade can at best be equal to the maximum grade $\hat{G}$ of a single bank, but, unlike the cumulative recovery, it can never be larger.

Relations between cumulative grade and cumulative recovery

From the above sums it is easy to derive that the cumulative grade and cumulative recovery are related:

$$m_F G_F R_{\Sigma}(n) = \sum_{i=1}^{n\leq N} m_{Ci}G_{Ci} = M_{\Sigma}(n)G_{\Sigma}(n) \tag{12}$$

We will now investigate if the product of the cumulative recovery minus the maximum recovery and the cumulative grade minus the maximum grade can be formed such that this product is independent of the cumulative grade and cumulative recovery. That is to say we form the product:

$$\left(R_\Sigma(n) - N\hat{R}\right)\left(G_\Sigma(n) - \hat{G}\right) = X \tag{13}$$

Where X is the constant sought. If X is independent of both the cumulative grade and the cumulative recovery, then this equation expresses that the cumulative grade versus cumulative recovery curve is a rectangular hyperbola with asymptotes $R_{max} = N\hat{R}$ and $G_{max} = \hat{G}$.

The distinction between a rectangular hyperbola and a normal hyperbola is important. A rectangular hyperbola has asymptotes that are orthogonal to each other. The right hand side of the equation of a hyperbola is then factorable into a product.

Substitute equation (12) into the hyperbola in equation (13) and expand the product will produce:

$$\frac{m_F G_F}{M_\Sigma(n)} R_\Sigma^{\ 2}(n) - R_\Sigma(n)\left(\hat{G} + N\hat{R}\frac{m_F G_F}{M_\Sigma(n)}\right) + N\hat{R}\hat{G} = X \tag{14}$$

The left hand side of equation (14) must be factorable into a product. In other words the left hand side polynomial of degree 2 must have a single root of multiplicity 2. This is only possible when the discriminant D of the polynomial is zero:

$$D = \left(\hat{G} + N\hat{R}\frac{m_F G_F}{M_\Sigma(n)}\right)^2 - 4N\hat{R}\hat{G}\frac{m_F G_F}{M_\Sigma(n)} = 0 \tag{15}$$

which after expanding and simplifying results into an equation for the cumulative mass flow of the concentrate:

$$M_\Sigma(n) = N\frac{\hat{R}}{\hat{G}} m_F G_F \tag{16}$$

This would imply that given a constant feed grade and feed mass flow, the mass flow in the concentrate is a constant. Obviously, this is a result of the inequality in equation (8). Reformulation of the latter in terms of a local maximum by using the actual bank number n gives an equation for the mass flow of the concentrate that is not constant and increases with the bank number n.

Back-substitution into equation (12) then gives a simple equality of proportions for the recovery and the grade:

$$\frac{R_\Sigma(n)}{N\hat{R}} = \frac{G_\Sigma(n)}{\hat{G}} \tag{17}$$

It must be stressed that the above two equalities in equation (16) and equation (17) are valid only because of the assumption made that the cumulative grade versus cumulative recovery curve is a hyperbola. This follows easily from the substation of the definition of the cumulative recovery in (15).

Note that the inequality in equation (8) is an absolute maximum valid for all bank numbers n and expressed in the maximum bank number N. This inequality may be reformulated with the help of equation (7). The inequality would then be formulated in terms of the actual bank number n. All of the results derived above remain valid when the maximum bank number *N* is replaced by the actual bank number *n*.

RESULTS AND DISCUSSION

In a paper by Neethling and Cilliers (2012), five sets of cumulative grade versus cumulative recovery are presented in the form of a graph. By digitizing this graph the coordinates of the cumulative grade and cumulative recovery points can be retrieved. The results of the digitizing are given in Table 1. The five different curves are labelled by the letters A to E. For reference the original symbols used in the graph of Neethling and Cilliers are given in the bottom row of the table.

Table 1 Digitized data points

Bank	A		B		C		D		E	
n	$R_\Sigma(n)$	$G_\Sigma(n)$	$R_\Sigma(n)$	$G_\Sigma(n)$	$R_\Sigma(n)$	$G_\Sigma(n)$	$R_\Sigma(n)$	$G_\Sigma(n)$	$R_\Sigma(n)$	$G_\Sigma(n)$
1	0.495	0.341	0.543	0.288	0.592	0.331	0.568	0.305	0.626	0.332
2	0.672	0.341	0.665	0.289	0.734	0.332	0.637	0.306	0.722	0.333
3	0.717	0.338	0.755	0.287	0.781	0.319	0.752	0.288	0.783	0.333
4	0.763	0.337	0.817	0.287	0.823	0.319	0.814	0.289	0.829	0.331
5	0.833	0.333	0.833	0.287	0.828	0.318	0.840	0.287	0.850	0.327
6	0.855	0.331	0.860	0.281	0.851	0.315	0.855	0.286	0.861	0.326
7	0.876	0.327	0.877	0.274	0.865	0.312	0.858	0.284	0.885	0.314
8	0.880	0.323			0.881	0.302	0.875	0.270	0.890	0.310
	Closed squares		Open squares		Circles		Diamonds		Triangles	

Raw cumulative grade cumulative recovery curves

By cross plotting the five sets of cumulative grade versus cumulative recovery in Table 1, we observe the typical shape of such curves with a monotonically increasing recovery towards an upper bound and a maximum grade.

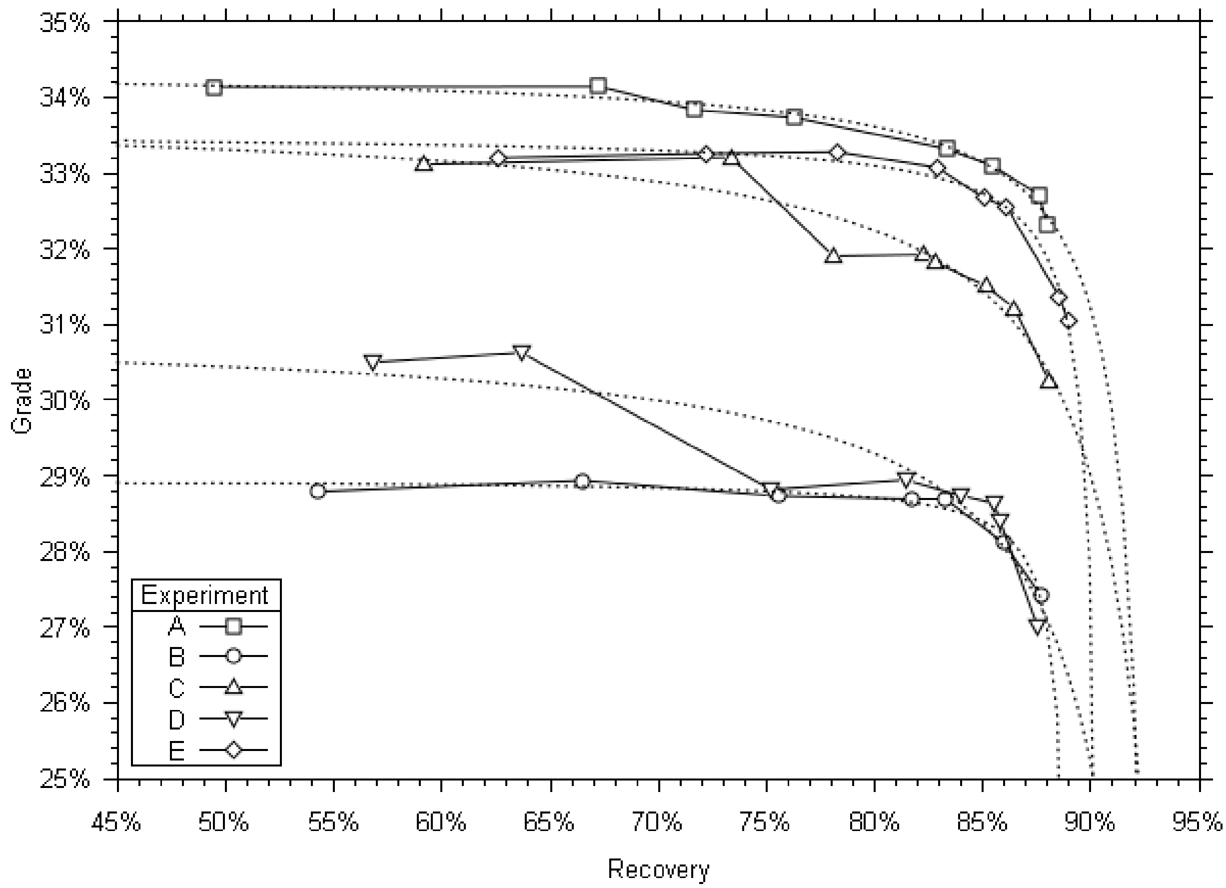

Figure 2 Grade recovery curve and curve fits

Normalized cumulative grade versus cumulative recovery curves.

The cumulative grade versus cumulative recovery curves of Neethling and Cilliers were fitted to a hyperbola with asymptotes A and B and constant C as follows:

$$(R-a)(G-b)=c \tag{18}$$

The fitted hyperbolas are shown in Figure 2 as dotted lines in the same color as the experimental data.

The result of the fit is given as the values of *a, b* and *c* where a is interpreted as the maximum attainable recovery R_{max}, *b* is interpreted as the maximum attainable grade G_{max} and *c* is interpreted as the performance coefficient *X*. Replotting the data from Neethling and Cilliers in terms of normalized grade ($G - G_{max}$) and normalized recovery ($R - R_{max}$) then produces a graph as in Figure 3, where all the cumulative grade and cumulative recovery curves share common asymptotes.

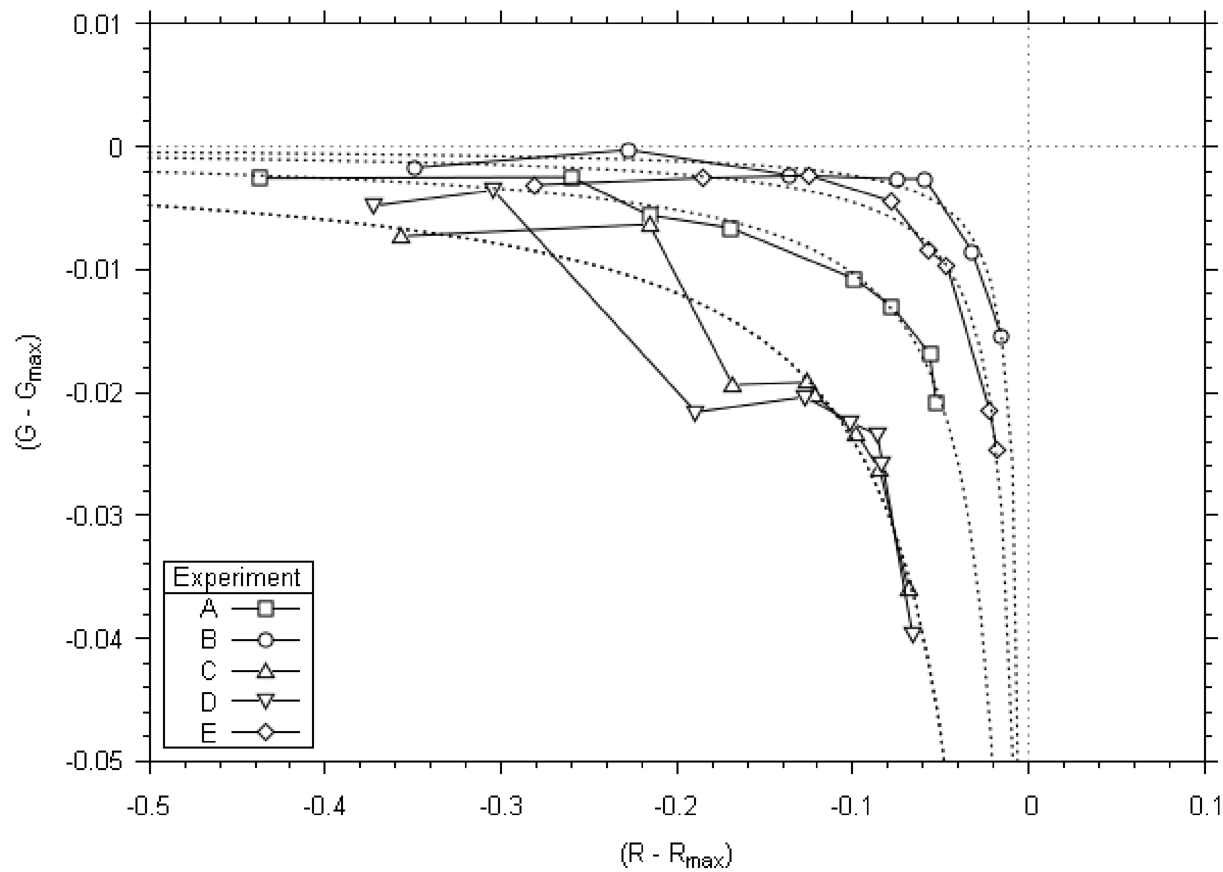

Figure 3 Normalized grade recovery curves

Observe the re-ordering in the curves. Whereas in Figure 2 A is the upper most one in Figure 3, after bringing all curves into a form with common asymptotes, B is now on top. Observe too that the C and D curves are now shown to be one and the same as is indicated by the fact that their fitted hyperbolae overlay.

Independent of the operating conditions of the cells in a bank of a row the maximum grade is determined by the mineralogy of the solids feed, i.e. the grade of the floatable material. Likewise, independent of the operating conditions of the cells in a bank of a row the maximum recovery is determined by the mass feed of solids. Thus, the normalization of the cumulative grade and cumulative recovery curves now shows the performance of the banks in a row independent of variations in the feed. Both any variation in mineralogy and variation in mass flow are eliminated. The normalized curves thus show, unbiased by uncontrollable parameters, the comparison in performance of the banks. The closer the normalized grade versus normalized recovery curve is to the asymptotes, the better the performance of the banks of cells in a row as this will ensure a high grade up to almost the maximum attainable recovery. In mathematical terms this means that the hyperbola's constant *c*, which equals the performance constant *X*, is small.

Reduced cumulative grade versus cumulative recovery curve

Reduction of the normalized grade and normalized recovery by the square root of the performance constant then leads to a uniform expression that is valid for all curves:

$$\left(\frac{R-R_{\max}}{\sqrt{X}}\right)\left(\frac{G-G_{\max}}{\sqrt{X}}\right)=1 \tag{19}$$

Since this equation equal a universal constant ("1") all the curves collapse to the same curve as is shown in Figure 4.

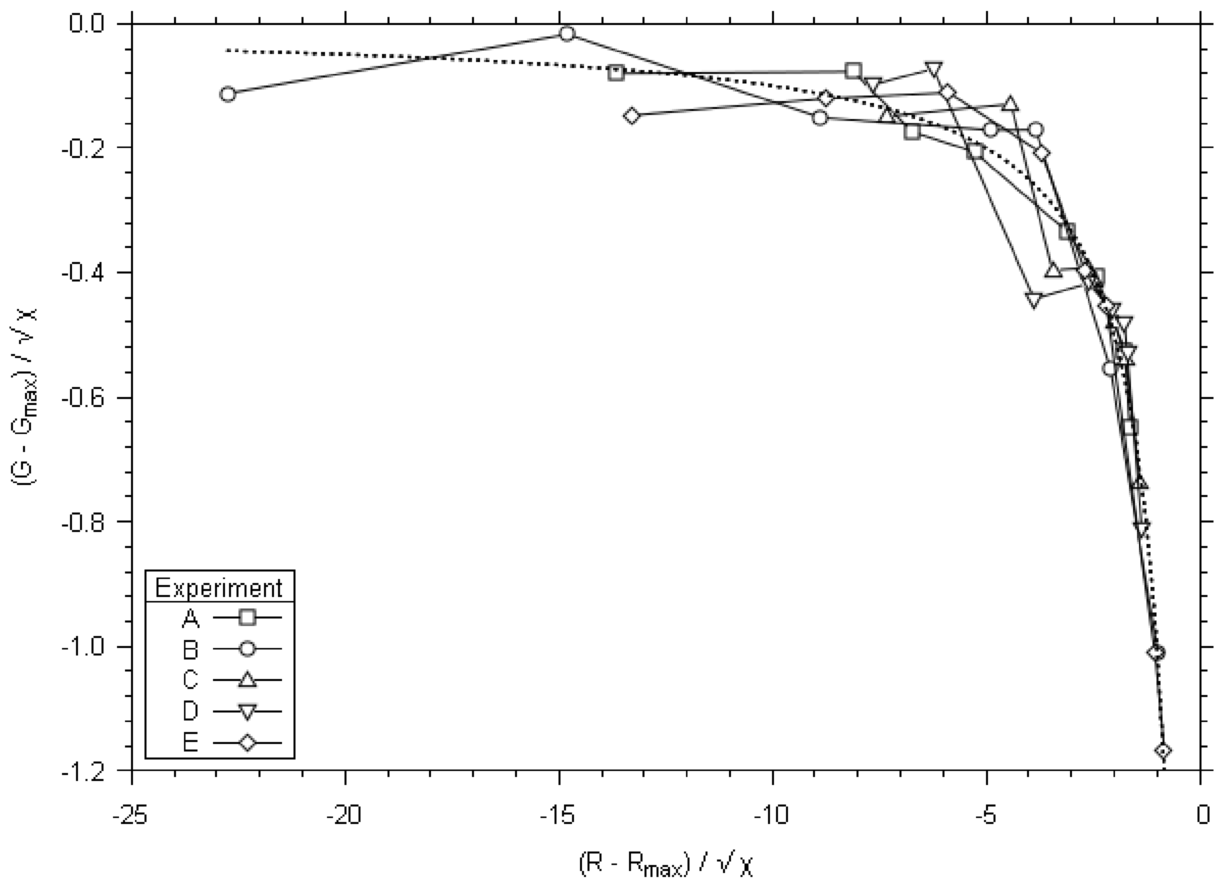

Figure 4 Reduced grade recovery curves

The reduced grade versus reduced recovery curve in Figure 4 shows to what extent the individual data points support the notion of a hyperbolic relation between grade and recovery. Observe how all the fitted hyperbolae now collapse to only one curve, whereas the experimental data points show some scatter around this universal curve. The scatter is inevitable given the nature of the experimental data. As is natural for accumulated data there is less scatter with increasing bank number i.e. with increasing reduced recovery.

The final result of the fitting are the individual fit parameters $a = R_{max}$, $b = G_{max}$ and $c = X$. These values have been compiled, along with their respective standard errors in Table 2. Indeed the experiment B, the green curve which tops the others in the normalized version of the cumulative grade versus cumulative recovery curves shows the lowest value of the performance coefficient, and hence the best performance

Table 2 Results, asymptotes and performance coefficient

Experiment	R_{max}		G_{max}		X	
	Fitted	StdError	Fitted	StdError	Fitted	StdError
A	0.932	0.017	0.344	0.002	0.0010	0.0004
B	0.892	0.006	0.290	0.001	0.0002	0.0001
C	0.949	0.038	0.338	0.006	0.0024	0.0016
D	0.941	0.056	0.310	0.008	0.0024	0.0023
E	0.907	0.005	0.335	0.001	0.0005	0.0001

Using the values of the maximum grade and the maximum recovery as per the individual fitted hyperbolae to each set of experimental data we may now evaluate the equality of ratios of grade and recovery as per equation (17). The cumulative mass flow in the concentrate, or cumulative mass pull, when normalized by the mass flow in the feed and the feed grade can be calculated as well with the help of equation (16). The results are compiled in Table 3.

Table 3 Grade recovery ratios and cumulative mass pull

Experiment	$R_\Sigma(N)/R_{max}$	$G_\Sigma(N)/G_{max}$	$M_\Sigma(N)/m_F G_F$	StdError
A	0.944	0.939	2.711	0.051
B	0.983	0.947	3.079	0.022
C	0.928	0.893	2.805	0.123
D	0.930	0.872	3.038	0.198
E	0.981	0.926	2.707	0.016

Perhaps not surprisingly it follows that the experiment B, which has the best performance in terms of the normalized cumulative grade versus normalized cumulative recovery curve, also shows the highest normalized cumulative mass pull. One might expect then that the experiment E would be second but this is not the case. Experiment D has a higher cumulative mass pull albeit with a large standard error.

The recovery and grade ratios should be equal according to equation (17) which follows from a mass balance equation inserted into the postulated hyperbolic shape of the cumulative grade versus cumulative recovery curve. The columns in Table 3 indicate to which extent this is supported by the experimental data.

CONCLUSION

By evaluation of inequalities of partial sums for cumulative grade and cumulative recovery, what is known in the industry is mathematically proven that: 1) Cumulative recovery increases with bank number. 2) Cumulative grade is independent of bank number.

The existence of a maximum cumulative grade and a maximum cumulative recovery combined with the experimentally observed shape of cumulative grade cumulative recovery curves leads us to postulate that such curves can be well described by rectangular hyperbolae.

Assuming that the cumulative grade-recovery curves are indeed rectangular hyperbolae, it is possible to derive an equation for the cumulative mass flow in the concentrate. Additionally, a simple equality of ratios for the cumulative grade and cumulative recovery can be derived.

Five sets of cumulative grade cumulative recovery data as published in the literature were digitized and fitted to hyperbolae. The fitting results in the values of the asymptotes were used to normalize the raw cumulative grade and raw cumulative grade curves. The normalized curves allow an unbiased performance evaluation of the row of banks with the performance factor, *X*, proving a metric for row performance.

Further reduction of the normalized cumulative grade versus normalized cumulative recovery curves collapses all five sets to one and the same common curve. The extent to which the data points indeed fall onto this single hyperbola can be used to gauge the validity of the assumptions made.

The normalized cumulative mass pull can be calculated based on the fitted values of the maximum recovery and maximum grade. This may serve as an alternative evaluation of row performance.

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